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Vector susceptibility and QCD phase transition in AdS/QCD models

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Abstract: We calculate the vector isospin susceptibility in AdS/QCD models to study QCD phase transition. In the hard wall model, we show explicitly that the infalling boundary condition at the horizon can be treated as a Dirichlet boundary with a fine-tuned boundary value in the zero frequency and momentum limit. With the infalling boundary condition, we uniquely determine the overall normalization of the vector isospin susceptibility in the hard wall model. In the framework of the soft wall model, we obtain the vector isospin susceptibility with and without Hawking-Page transition and compare our results with lattice QCD. We briefly discuss the chiral symmetry restoration in the AdS/QCD models.

Keywords: Gauge-gravity correspondence, QCD.

Contents

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1. Introduction

There has been much interest in applying the idea of AdS/CFT [1] in strong interaction. After initial set up for N=4 SYM theory, confining theories were treated with IR cut off at the AdS space [2] and quark flavors [3] were introduced by adding extra probe branes. More phenomenological models were also suggested to construct a holographic model dual to QCD [4−7]. The finite temperature version of such model were worked out in [8, 9]. However, the meson spectrum of the model does not follow the well known Regge trajectories. To remedy this symptom of the these models, quadratic dilaton background was introduced ref. [10] whose role is to prevent string going into the deep inside the IR region of AdS space and as a consequence the particle spectrum rise more slowly compared with hard-wall cutoff. Remarkably such a dilaton-induced potential gives exactly the linear trajectory of the meson spectrum. In [11], it was pointed out that such a dilatonic potential could be motivated by instanton effects.

The vector isospin susceptibility χ_V, which measures the response of QCD to a change in the chemical potential, was proposed as a probe of the QCD chiral/deconfinement phase transition at zero chemical potential [12, 13]. It is one of the thermodynamic observables that can reveal a character of chiral phase transition. The lattice QCD calculation [12] showed the enhancement of χ_V around T_c by a factor 4 or 5. Since then, various model studies [14 − 16] and lattice simulations [17 − 22] have been performed to calculate the susceptibility.

In this work, we calculate the vector isospin susceptibility, χ_V, and study QCD chiral/deconfinement phase transition in holographic QCD models [6 − 8, 10]. In the models adopted in the present work, we implicitly assume that chiral symmetry restoration and deconfinement take place at the same critical temperature T_c.

In a gravity dual of QCD-like model, the confinement to de-confinement phase transition is described by the Hawking-Page transition (HPT). At low temperature, thermal AdS dominates the partition function, while at high temperature, AdS-black hole geometry dominates. This was first discovered in the finite volume boundary case in [23], and more recently it is shown in [24] that the same phenomena happen also for infinite boundary volume, if there is a finite scale along the fifth direction. In our work, both hard wall [6] and soft wall [10] models are considered, where we have definite IR scale, so that we are dealing with theories with deconfinement phase transition.

In the presence of the AdS black hole, the most physical boundary condition is the infalling one. We show explicitly in the hard wall model [6, 7] that the infalling boundary condition in the zero frequency and momentum limit can be understood as a Dirichlet condition whose field value is special. The value is fine-tuned one such that field configuration is a limit of infalling wave with zero frequency and momentum. If we are not interested in a overall temperature independent normalization constant, this observation simplifies the analysis greatly. In the the soft wall model [10], we calculate the vector isospin susceptibility and find that the lattice QCD data is in between our results with and without the HPT. The back reaction of the metric to the matter field could be very important in discussing the phase transition, which however is not within the scope of this paper.

The rest of the paper goes as follows. In section 2, we briefly summarize the holographic QCD models adopted in the present work and discuss the chiral symmetry restoration in the models. In section 3, we first calculate the vector susceptibility in the AdS black hole background followed by the implication of a Hawking-Page transition [24]. Section 4 gives summary.

2. Holographic QCD at finite temperature

Hard wall model: The action of the holographic QCD model suggested in [6 − 8] is given by

$$\begin{aligned} \mathrm{S_I} &= \int d^4x dz \sqrt{g}\mathcal{L}_5 \, , \\ \mathcal{L}_5 &= \mathrm{Tr}\left[-\frac{1}{4g_5^2}(L_{MN}L^{MN} + R_{MN}R^{MN}) + |D_M\Phi|^2 - M_\Phi^2|\Phi|^2 \right] , \end{aligned} \tag{2.1}$$

where $D_M\Phi = \partial_M\Phi - iL_M\Phi + i\Phi R_M$ and $L_M = L_M^a\tau^a/2$ with τ^a being the Pauli matrix. The scalar field is defined by $\Phi = S\mathrm{e}^{i\pi^a\tau^a}$ and $< S > \equiv \frac{1}{2}v(z)$, where S is a real scalar and π is a pseudoscalar. Under $\mathrm{SU}(2)_V$, S and π transform as singlet and triplet. In this model, the 5D AdS space is compactified such that $0 < z \leq z_m$, where z_m is a infrared (IR) cutoff, which is fixed by the rho-meson mass at zero temperature: $1/z_m \simeq 320$ MeV,

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and the value of the 5D gauge coupling g_5^2 is identified as $g_5^2 = 12\pi^2/N_c$ through matching with QCD [6, 7, 10].

As in [8], we work on the 5D AdS-Schwarzchild background, which is known to describe the physics of the finite temperature in dual 4D field theory side,

$$ds_5^2 = \frac{1}{z^2}\left(f(z)dt^2 - (dx^i)^2 - \frac{1}{f(z)}dz^2\right), \quad f(z) = 1 - \frac{z^4}{z_T^4}, \tag{2.2}$$

where $i = 1, 2, 3$. Here the temperature is defined by $T = 1/(\pi z_T)$ and the critical temperature T_c for deconfinement is given as $T_c = 1/(\pi z_m)$ [8]. We note here that an analysis based on a Hawking-Page type transition [24] in the hard wall model predicts $T_c = 2^{1/4}/(\pi z_m) \simeq 1.189/(\pi z_m)$. With the hard wall model, if IR cutoff z_m exists outside the horizon ($z_m < z_T$) it corresponds to the confined phase of QCD, while the opposite case ($z_m > z_T$) describes the deconfined phase [8]. The equation of motion for $v(z)$ is

$$\left[\partial_z^2 - \frac{4-f}{zf}\partial_z + \frac{3}{z^2 f}\right]v(z) = 0\,, \tag{2.3}$$

and the solution is obtained by [8]

$$v(z) = M_q z\ {}_2F_1\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{z^4}{z_T^4}\right) + \Sigma_q z^3\ {}_2F_1\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}, \frac{z^4}{z_T^4}\right). \tag{2.4}$$

Here M_q and Σ_q are identified with the current quark mass and the chiral condensate respectively. Note that at $z = z_T$, both terms in $v(z)$ diverges logarithmically. To avoid this, the analysis in ref. [8] is done only for confined phase where the horizon is behind the cut off.

Our point here is to discuss the nature of the deconfined phase $T \to T_c$. We note here that the divergence as (i.e, $z_T \to z_m$) requires us to set both of them to be zero,

$$M_q = 0, \qquad \Sigma_q = 0. \tag{2.5}$$

Now we interpret the absence of the chiral condensation as the signal of the chiral symmetry restoration in this specific model. The fact that chirally symmetric phase do not allow current quark mass is logically correct. In reality the the chiral symmetry is partially broken in low temperature and also partially restored in the high temperature, since current quark mass is still not completely zero. In this sense, the hard wall model respect the chiral symmetry more than the reality.[1]

Soft wall model: In [10], dilaton background was introduced for the Regge behavior of the spectrum and we work mostly in this frame work.

$$\mathrm{S_{II}} = \int d^4x dz \mathrm{e}^{-\Phi}\mathcal{L}_5\,, \tag{2.6}$$

where $\Phi = cz^2$. Here the role of the hard-wall IR cutoff z_m is replaced by dilaton-induced potential. The equation of motion for $v(z)$ is given by

$$\left[\partial_z^2 - \left(2cz + \frac{4-f}{zf}\right)\partial_z + \frac{3}{z^2 f}\right]v(z) = 0\,. \tag{2.7}$$

[1] In string theory models like D3-D7 system, Σ_q, M_q are not necessarily zero even for deconfined phase. In that sense, the chiral symmetry in D3-D7 system is not very well defined.

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At zero temperature [10], where $f = 1$, one of two linearly independent solutions of eq. (2.7) diverges as $z \to \infty$, and so we have to discard this solution. Then chiral condensate is simply proportional to M_q [10].

Now we ask what happens at finite temperature. Near z_T, solution of eq. (2.7) has following behavior:

$$v(z) \sim c_1 + c_2 \ln(1 - z/z_T) \text{ as } z \to z_T\,. \tag{2.8}$$

Again one of the solutions diverges near z_T so that we have to abandon c_2 part. Then we have only one input parameter so that M_q and Σ_q are not independent. Therefore if we "impose" chiral symmetry to be restored, M_q and Σ_q should vanish simultaneously. This is similar to the hard wall model. However, the soft wall model does not say that chiral symmetry should be restored at high temperature, while the hard wall model does.

3. Susceptibilities

In this section, we calculate the vector isospin susceptibility in those models to study QCD chiral/deconfinement phase transition. The quark number susceptibility was proposed as a probe of the QCD chiral phase transition at zero chemical potential [12, 13]. The flavor singlet and non-singlet susceptibilities, χ_S and χ_{NS} respectively, are defined by

$$\chi_S = \left(\frac{\partial}{\partial\mu_u} + \frac{\partial}{\partial\mu_d}\right)(n_u + n_d), \quad \chi_{NS} = \left(\frac{\partial}{\partial\mu_u} - \frac{\partial}{\partial\mu_d}\right)(n_u - n_d)\,, \tag{3.1}$$

where μ_i is the chemical potential and n_i is the quark number density with $i = u, d$. Note, however, that $\chi_S \simeq \chi_{NS}$ [12, 17, 18, 21]. In this work, we consider the flavor non-singlet susceptibility with the normalization convention in ref. [16]. The vector and axial-vector susceptibility $\chi_V(T)$ and $\chi_A(T)$ for non-singlet currents are given by the 00-component of the vector and axial current-current correlators respectively in the zero energy and zero momentum limit:

$$\chi_{V,A}(T) = 2N_f \lim_{\vec{p}\to 0} \lim_{p_0\to 0} \left[G^{00}_{V,A}(p_0, \vec{p}; T)\right] \tag{3.2}$$

where $2N_f$ is included as a normalization factor. The correlators at finite temperature (in imaginary time formalism) are defined by

$$G^{\mu\nu}_{V,A}(p_0 = i\omega_n, \vec{p}; T)\delta_{ab} = \int_0^{1/T} d\tau \int d^3\vec{x} e^{-i(\vec{p}\cdot\vec{x}+\omega_n\tau)} \left\langle J^\mu_{V,Aa}(\tau,\vec{x}) J^\nu_{V,Ab}(0,\vec{0})\right\rangle_\beta \tag{3.3}$$

where $\omega_n = 2n\pi T$ is the Matsubara frequency.

From [14], the density-density correlation which is nothing but the 0-0 component of the vector-vector correlations or fluctuations

$$\chi_q(T, \mu_q) = \beta \int dx G^V_{oo}(0, x), \tag{3.4}$$

where G^V_{00} is defined in eq. (3.3). After Fourier transformation, it is rewritten

$$\chi_q(T, \mu_q) = \lim_{k\to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^V_{oo}(\omega, k) \tag{3.5}$$

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By the fluctuation-dissipation theorem yields,

$$G^V_{00}(\omega,k) = -\frac{2}{1-\mathrm{e}^{-w/T}}\mathrm{Im}G^R_{00}(\omega,k) \tag{3.6}$$

where $\Pi^R_{00}(\omega,k)$ is retarded Greens function of $j_m u, j_n u$.

$$\begin{aligned}\chi_q(T,\mu_q) &\equiv \frac{\partial\rho_q}{\partial\mu_q}\\ &= -\lim_{k\to 0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{2}{1-\mathrm{e}^{-w/T}}\mathrm{Im}G^R_{00}(\omega,k)\end{aligned} \tag{3.7}$$

This $\mathrm{Im}G^R_{00}(\omega,k)$ is related to the real part of Greens function, applying the Kramers-Kronig dispersion relations,

$$\mathrm{Re}G^R_{00}(\omega,k) = \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega\prime}{\pi}\frac{\mathrm{Im}G^R_{00}(\omega,k)}{w\prime - w} \tag{3.8}$$

then one can see that χ_q is written concisely in the following form,

$$\chi_q(T,\mu_q) = -\lim_{k\to 0}\mathrm{Re}G^R_{00}(0,k) \tag{3.9}$$

The lattice QCD calculations [12, 17, 18, 21] showed the enhancement of χ_V around T_c:

$$R_\chi \equiv \frac{\chi_V(T_c+\epsilon)}{\chi_V(T_c-\epsilon)} = 4\sim 5\,. \tag{3.10}$$

The enhancement in R_χ may be understood roughly as follows [13]. At low temperature, in confined phase χ_V will pick up the Boltzmann factor $\mathrm{e}^{-M_N/T}$, where M_N is a typical hadron mass scale $\sim$ 1 GeV, while at high temperature the factor will be given in terms of a quark mass $\mathrm{e}^{-M_q/T}$, and therefore there could be some enhancement. In ref. [14], it is shown that the enhancement in χ_V may be due to the vanishing or a sudden decrease of the interactions between quarks in the vector channel.

In the holographic QCD models, due to the HPT [24], the vector isospin susceptibility is described by the AdS black hole background at high temperature regime and by the thermal AdS at low temperature regime. We will patch them together. We calculate the vector isospin susceptibility in the hard wall model [6, 7] and in the soft wall model [10] without Hawking-Page transition first with an assumption that AdS black hole is stable even at low temperature, and then we describe how HPT changes the picture.

3.1 Hard wall model

We consider the infalling boundary condition. In the presence of the AdS black hole, especially when the horizon is inside of the wall ($z_T < z_m$), the most natural boundary condition (at least for wave like solution) is the infalling boundary condition, since the black hole can only absorb classically. To impose the infalling boundary condition, we solve the problem at small but non-zero frequency and momentum and then take them to be zero. This is the problem considered in the literature on hydrodynamics [26]. Here we

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give a comparison between the Dirichlet and Infalling boundary in the hard wall model and give a sketch of the calculation in appendix A. First we write the solution as sum of two independent solutions: one infalling and the other outgoing in terms of a dimensionless coordinate $u = (z/z_T)^2$;

$$V_0(u) = aY_{k,in}(u) + bY_{k,out}(u), \tag{3.11}$$

and set the infalling BC

$$b = 0, \quad a = 1 \tag{3.12}$$

and then take $k = (\omega, q)$ to zero with ω first. Here we assume that Y's are normalized to be 1 at the boundary. It is known [26] that near horizon ($u = 1$) behavior of the wave function is

$$Y_{k,in}(u) \sim (1-u)^{-i\omega/2\pi T}(1 + d_1 u + d_2 u^2 + \cdots) \tag{3.13}$$

and all the temperature dependence comes in the combination of $\omega/2\pi T$ or $q/2\pi T$. Therefore in the limit of zero frequency and momentum, there is no temperature dependence of the coefficient d_i. Furthermore, since Y's should be reduced to the static solution in zero frequency and zero momentum limit, the infalling boundary condition is effectively equivalent to

$$V_0(z) = \lim_{k\to 0} Y_{k,in}(u) = 1 + c_2(z/z_T)^2, \tag{3.14}$$

where $c_2 = Y'_{0,in}(u = 0)$ is a number that is already determined. The fact that c_2, a temperature independent constant, is determined not by hand but by the physical condition is the unique feature of the infalling boundary condition compared to the Dirichlet condition. In other words, *for the zero frequency and zero momentum, the infalling boundary condition can be effectively described by a Dirichlet boundary condition where the IR boundary value is taken as a fine-tuned one.*

The vector isospin susceptibility in this case is that

$$\chi_V(T) = 2N_f \frac{-2c_2}{g_5^2}\pi^2 T^2. \tag{3.15}$$

In fact, using the result of ref. [26], one can determine c_2 as follows. One should notice that in terms of the rescaled coordinate $u = (z/z_T)^2$, the momentum $k = (\omega, 0, 0, q)$ enters in the action only in the combination of $\mathfrak{w} = \frac{\omega}{2\pi T}, \quad \mathfrak{q} = \frac{q}{2\pi T}$. The relevant equations of motion for the vector fields in the $A_u = 0$ gauge are

$$\mathfrak{w}A'_0 + \mathfrak{q}f\,A'_3 = 0\,, \quad A''_0 - \frac{1}{uf}\left(\mathfrak{q}^2 A_0 + \mathfrak{w}\mathfrak{q}A_3\right) = 0\,, \tag{3.16}$$

$$A''_3 + \frac{f'}{f}A'_3 + \frac{1}{uf^2}\left(\mathfrak{w}^2 A_3 + \mathfrak{w}\mathfrak{q}A_0\right) = 0\,, \tag{3.17}$$

where $'$ means ∂_u. Out of the coupled equations we can eliminate A_3 to get a second order differential equation for $A'_0 := \Psi$.

$$\Psi'' + \frac{(uf)'}{uf}\Psi' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f(u)}{uf^2}\Psi = 0\,, \tag{3.18}$$

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After taking out the near horizon behavior $\sim (1-u)^{-i\mathfrak{w}/2}$ of infalling wave, one can extract the small frequency behavior of the residual part using Mathematica. The result is

$$\Psi = (1-u)^{-i\mathfrak{w}/2} \cdot \frac{\mathfrak{q}^2 A_0^0 + \mathfrak{w}\mathfrak{q} A_3^0}{(i\mathfrak{w} - \mathfrak{q}^2)} \left(1 + \frac{i\mathfrak{w}}{2} \ln \frac{2u^2}{1+u} - \mathfrak{q}^2 \ln \frac{2u}{1+u} + \cdots \right), \tag{3.19}$$

where A_μ^0 is the boundary value of A_μ. With the prescription for the retarded Green function

$$G^R_{\mu\nu}(k) = \frac{\delta^2 S}{\delta A^0_\mu(-k)\delta A^0_\nu(k)}, \tag{3.20}$$

with

$$S = \frac{\pi^2 T^2}{g_5^2} \int d^4k \Big[\Psi(-k)\Psi(k) + \cdots \Big]_{u=0}. \tag{3.21}$$

one can get the retarded Green function

$$G^R_{\mu\nu}(\omega, \boldsymbol{q}) = -i \int d^4x \, e^{-iq\cdot x}\, \theta(t) \langle [J_\mu(x),\, J_\nu(0)] \rangle \tag{3.22}$$

in small frequency;

$$G^R_{00} = \frac{2\pi^2 T^2}{g_5^2} \frac{\mathfrak{q}^2}{(i\mathfrak{w} - \mathfrak{q}^2 + i\mathfrak{w}^2 \ln 2/2)} \left(1 + \ln 2 \left(\frac{i}{2}\mathfrak{w} - \mathfrak{q}^2\right)\right) \tag{3.23}$$

Finally, by using eq. 3.9, we get the susceptibility

$$\chi_V = 2N_f \frac{2\pi^2 T^2}{g_5^2}, \tag{3.24}$$

from which one can read off the c_2.

$$c_2 = -1 \tag{3.25}$$

In summary, the infalling and Dirichlet boundary conditions give the same result on the vector isospin susceptibility apart from the fact that the infalling condition at the horizon uniquely fixes the overall normalization. The vector isospin susceptibility in eq. (3.24) is rather too small compared to the results from model studies [14 − 16] and lattice simulations [17 − 22], where $c_2 \sim -0.8$.

3.2 Soft wall model

Now we consider the soft wall model [10]. By imposing infalling boundary condition. Following the same procedure given in section 3.1, we arrive at[2]

$$\chi_V = N_f \frac{\pi T^2}{g_5^2} \frac{\mathrm{e}^{-c_T}}{3} \Big(-2 + 2\mathrm{e}^{c_T} + 3(c_T - 2)\mathrm{e}^{2c_T} (Ei(-2c_T) - Ei(-c_T)) \Big), \tag{3.26}$$

which is plotted in figure 3.2. Note that due to the HPT, we can trust the figure only when $\tilde{T}/T_c \geq 1$.[3] As in the hard wall model, χ_V is too small with the infalling boundary condition also in the soft wall model.

[2] YK: Just for our discussion, I included Kwang-Hyun's note in appendix B.

[3] YK: For our discussion I showed low temperature regime too in figure 2. Later, we may plot only $\tilde{T}/T_c \geq 1$. In figure 1 and 2, x-axis label will be corrected as $\tilde{T}/T_c$.

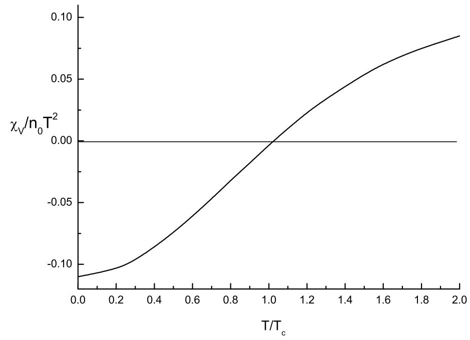

Figure 1: χ_V in the soft wall model with the infalling boundary condition.

Finally, we give a brief comment on the axial-vector susceptibility. For this we consider the axial vector A_0. The equation of motion reads

$$\left[\partial_z^2 - \frac{1}{z}\partial_z + \frac{\vec{q}^2}{f(z)} - g_5^2 \frac{1}{z^2 f} v(z)^2\right] A_0(z, \vec{q}) = 0\,. \tag{3.27}$$

For the chiral symmetric regime, the AdS/background is not relevant while in the chiral symmetry restored regime, $v(z)$ becomes zero. Then eq. (C.1) and eq. (3.27) become identical, implying $\chi_A(T_c) = \chi_V(T_c)$.[4]

4. Summary

We first discussed the chiral symmetry restoration in AdS/QCD models. The AdS/QCD models respect the chiral symmetry more rigidly than the reality in the sense that, in chiral symmetry restored phase, both of the chiral condensate and the mass of the quarks are zero.

Then, we calculated the vector isospin susceptibility in both hard wall and soft wall models. In the hard wall model, we show that, for the zero frequency and zero momentum, the infalling boundary condition can be effectively described by a Dirichlet boundary condition with a fine-tuned IR boundary value. For the hard wall model with the HPT, we observe that the vector isospin susceptibility jumps from 0 to a finite value. For the soft wall model, our result with no HPT agrees qualitatively with the results from various model calculations and lattice QCD. With the HPT, we predict that $\chi_V = 0$ at low temperature from the finiteness of V_0 at IR to guarantee 5D action to be finite, and we also obtain the temperature dependence of χ_V at high temperature apart from the overall normalization that is fixed by a IR boundary condition. Our result with the HPT exhibits a similar behavior observed in model studies [14 − 16] and lattice simulations [17 − 22]. Our results in both models predict a sharp jump in the vector isospin susceptibility, which is the unavoidable aspect of the HPT and could be smoothed out by including large N_c corrections.

Finally, we discuss a limitation of our approach in the light of the QCD phase transition. The nature of the QCD transition depends on the number of quark flavors and the quark

[4] One may turn this statement around and argue that $\chi_A(T_c) = \chi_V(T_c)$ implies $v(z) = 0$ near T_c.

mass: for pure SU(3) gauge theory, it is a first order, for two massless quarks, it is a second order, for two quarks with finite masses, it is a cross over, for three degenerate massless quarks, it is a first order, etc. Unlike Polykov loop or the chiral condensate, the vector isospin susceptibility is not an order parameter, and so in the present study we are not able to determine the order of the QCD phase transition. The vector isospin susceptibility could serve, at best, as an indicator of the transition.

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A. Estimation of c of soft wall model.

Now let's estimate the value of c and $\tilde{T}_c$. The masses of the vector mesons are given by

$$m_n^2 = 4c(n+1)\,. \tag{A.1}$$

If we use $m_1 = 770$ MeV and $m_2 = 1450$ MeV to calculate the slope of the Regge trajectory, then we obtain $\sqrt{c} \simeq 614$ MeV and so end up with the reasonable value of the transition temperature $T_c \simeq 195$ MeV.[5] We note here that in [30], the value of $\sqrt{c}$ was determined to be ~ 671 MeV. Finally, we relate c with the QCD string tension σ. The masses of vector towers are given, in terms of σ, by

$$m_n^2 = 2\pi\sigma n\,. \tag{A.2}$$

Comparing the slope of the Regge trajectory in (A.1) and (A.2), we find out

$$c = \frac{\pi}{2}\sigma\,, \tag{A.3}$$

and so the dilaton factor becomes $\mathrm{e}^{-\frac{\pi}{2}\sigma z^2}$. Note that the relation between c and string tension was also observed in ref. [32].

B. Infalling boundary condition in the soft wall model

From the recipe of [26], rescale coordinate z as u=$(z/z_T)^2$. Then the metric is

$$ds^2 = \frac{(\pi T)^2}{u}\Big(f(u)dt^2 - (dx^i)^2\Big) - \frac{1}{4u^2 f(u)}du^2 \tag{B.1}$$

[5] Recently, the value of T_c was determined in ref. [29], $T_c = 210$ MeV, and very recently, the relation between T_c and z_m ($\sqrt{c}$) is obtained through Hawking-Page analysis in the holographic models used in this work, where $T_c \approx 191$ MeV [24].

where f(u) = 1-u^2 and T = $1/\pi z_T$. We use the soft wall model [10], the action is given

$$S = \int d^5x\sqrt{g}\mathrm{e}^{-\Phi}\frac{1}{4}F^2, \quad \Phi = \frac{c}{(\pi T)^2}u. \tag{B.2}$$

After Fourier transformation and rescaling energy and momentum in unit of temperature, $\mathfrak{w} = w/(2\pi T), \mathfrak{q} = q/(2\pi T)$, resulting equations of motion for each fields are

$$A_t: \qquad A_t'' - \frac{c}{(\pi T)^2}A_t' - \frac{1}{uf(u)}(\mathfrak{q}^2A_t + \mathfrak{q}\mathfrak{w}A_z) = 0 \tag{B.3}$$

$$A_\alpha: \qquad A_\alpha'' + \left(\frac{-c}{(\pi T)2} + \frac{f'(u)}{f(u)}\right)A_\alpha' + \frac{1}{uf(u)}\left(\frac{\mathfrak{w}^2}{f(u)} - \mathfrak{q}^2\right)A_\alpha = 0 \tag{B.4}$$

$$A_z: \qquad A_z'' + \left(\frac{-c}{(\pi T)2} + \frac{f'(u)}{f(u)}\right)A_z' + \frac{1}{uf(u)^2}(\mathfrak{w}\mathfrak{q}A_t - \mathfrak{w}^2A_z) = 0 \tag{B.5}$$

$$A_u: \qquad \mathfrak{w}A_t' + \mathfrak{q}f(u)A_z' = 0 \tag{B.6}$$

From the eq. (B.3), we can express A_z in terms of A_t

$$A_z = \frac{uf(u)}{\mathfrak{q}\mathfrak{w}}\left(A_t'' - \frac{c}{(\pi T)^2}A_t' - \frac{\mathfrak{q}^2}{uf(u)}A_t\right) \tag{B.7}$$

and by substituting it to eq. (B.6), equation is decoupled.

$$A''' + \left(\frac{1-3u^2}{uf} - c_T\right)A'' + \frac{1}{uf^2}\Big(\mathfrak{w}^2 - f(c_T(1-3u^2) + \mathfrak{q}^2)\Big)A' = 0 \tag{B.8}$$

where $c_T = c/(\pi T)^2$. The natural boundary condition of black hole is infalling. Imposing infalling boundary condition at black hole horizon is $A_t' = (1-u)^{-i\mathbf{w}/2}$ F(u). Then eq. (B.8) is

$$\begin{aligned} &F'' + \left(-\frac{c}{(\pi T)^2} + \frac{1-3u^2}{uf(u)} + i\frac{\mathbf{w}}{1-u}\right)F' \\ &+\Bigg(-\frac{c}{(\pi T)^2}\left(\frac{1-3u^2}{uf(u)}\right) + \mathfrak{w}\left(\frac{i(1+2u)}{2uf(u)} - \frac{c}{(\pi T)^2}\frac{i(1+u)}{2f(u)}\right) \\ &\quad +\mathfrak{w}^2\frac{4-u(1+u)^2}{4uf(u)^2} - \frac{\mathfrak{q}^2}{uf(u)}\Bigg)F = 0. \end{aligned} \tag{B.9}$$

In the long-wavelenghth, low frequency limit, F(u) can be expanded as series in $\mathfrak{w}, \mathfrak{q}^2$,

$$F(u) = F_0 + \mathfrak{w}F_1(u) + \mathfrak{q}^2G_1 + \cdots \tag{B.10}$$

After some algebra, we get the first few terms

$$F_0 = B \tag{B.11}$$

$$\begin{aligned} F_1 = \frac{iB\mathrm{e}^{-c_T}}{12}\Big(&2(\mathrm{e}^{c_T} - \mathrm{e}^{c_Tu}) + (-12+5c_T)\mathrm{e}^{c_T(1+u)}\Big(Ei(-c_T) - Ei(-c_Tu)\Big)\rangle \\ &-2\mathrm{e}^{c_T(2+u)}(-3+c_T)\{Ei(-2c_T) - Ei(-c_T(1+u))\}\Big) \end{aligned} \tag{B.12}$$

$$G_1 = Be^{c_T u}\Big(Ei(-c_T) - Ei(-c_T u) + e^{c_T}\Big(Ei(-c_T(1+u)) - Ei(-2c_T)\Big)\Big) \quad \text{(B.13)}$$

where Ei(x) is 'ExpIntegralEi(x)', defined as the principal value of $Ei(z) = -\int_{-z}^{\infty} e^{-t}/t\,dt$. The integration constants of F_1, G_1 is chosen by the regularity condition at u=1. And B is determinded from the boundary value of A_t, A_z at u=0.

$$\lim_{u\to 0} A_t(u) = A_t^0, \qquad \lim_{u\to 0} A_z(u) = A_z^0$$
$$B = \frac{\mathfrak{q}^2 A_t^0 + \mathfrak{q}\mathfrak{w} A_z^0}{i\mathfrak{w}(1-\frac{5}{12}c_T) - \mathfrak{q}^2}. \quad \text{(B.14)}$$

From these result, $A_t'(u)$ is

$$A_t'(u) = (1-u)^{-i\mathrm{w}/2}\frac{\mathfrak{q}^2 A_t^0 + \mathfrak{q}\mathfrak{w} A_z^0}{i\mathfrak{w}(1-\frac{5}{12}c_T) - \mathfrak{q}^2}\left(1 + \mathfrak{q}^2 X_{\mathfrak{q}^2} + i\frac{\mathfrak{w}}{12}Y_{\mathfrak{w}}\right) \quad \text{(B.15)}$$

where $X_{\mathfrak{q}^2}, Y_{\mathfrak{w}}$ are

$$\begin{aligned} X_{\mathfrak{q}^2} &= e^{c_T u}\Big(Ei(-c_T) - Ei(-c_T u) - e^{c_T}\Big(Ei(-2c_T) - Ei(-c_T(1+u))\Big)\Big) \\ Y_{\mathfrak{w}} &= e^{c_T u}\Big(2e^{-c_T u} - 2e^{-c_T} + (-12+5c_T)\Big(Ei(-c_T) - Ei(-c_T u)\Big) \\ &\quad -2e^{c_T}(-3+c_T)\Big(Ei(-2c_T) - Ei(-c_T(1+u))\Big)\Big) \end{aligned} \quad \text{(B.16)}$$

This A_t is related to the retarded Green function, because we choose infalling boundary condition at horizon,

$$G^R = A(u) f_{-q}(u)\partial_u f_q(u)|_{u\to 0}. \quad \text{(B.17)}$$

$A_t'(u\to 0)$ is

$$A_t'(\epsilon) = \frac{\mathfrak{q}^2 A_t^0 + \mathfrak{q}\mathfrak{w} A_z^0}{i\mathfrak{w}(1-\frac{5}{12}c_T) - \mathfrak{q}^2}\left(1 + \mathfrak{q}^2 X_{\mathfrak{q}^2} + i\frac{\mathfrak{w}}{12}Y_{\mathfrak{w}}\right) \quad \text{(B.18)}$$

where $X_{\mathfrak{q}^2}(\epsilon), Y_{\mathfrak{w}}(\epsilon)$ are

$$\begin{aligned} X_{\mathfrak{q}^2}(\epsilon) &= Ei(-c_T) - Ei(-\epsilon) - e^{c_T}\Big(Ei(-2c_T) - Ei(-c_T)\Big) \\ Y_{\mathfrak{w}}(\epsilon) &= 2 - 2e^{-c_T} + (-12+5c_T)\Big(Ei(-c_T) - Ei(-\epsilon)\Big) \\ &\quad -2e^{c_T}(-3+c_T)\Big(Ei(-2c_T) - Ei(-c_T)\Big) \end{aligned} \quad \text{(B.19)}$$

The retarded Green function is

$$G_{oo}^R = \frac{2\pi^2 T^2}{g_5^2} A_t'(\epsilon). \quad \text{(B.20)}$$

where A_t' has only A_t^0 part.

$$\mathrm{Re}G_{00}^R(k) = -\frac{2\pi^2 T^2}{g_5^2}\frac{q^2}{P^2 w^2 + \bar{D}^2 q^4}\left(\bar{D}^2 q^2 + \bar{D}^4 q^4 X_{\mathfrak{q}^2}(\epsilon) - \frac{w^2}{12}\bar{D}^2 P Y_{\mathfrak{w}}(\epsilon)\right) \quad \text{(B.21)}$$

this $\bar{D} = 1/2\pi T$. Note that this Greens function has diffusion pole at $P^2 w^2 + \bar{D}^2 q^4 = 0$. Then diffusion constant D is given as

$$D = \frac{\bar{D}}{P} = \frac{1/2\pi T}{1 - \frac{5}{12}\frac{c}{\pi^2 T^2}} \tag{B.22}$$

Is this diffusion constant positive definite? From [24], there is a phase transition so called Hawking-Page transition and the critical temperature is at $cz_h^2 = 0.419035$. So below that temperature, thermal adS is dominant. If we consider black hole phase, we should be aware of $T > T_C$. In this case, this diffusion constant is positive definite

$$1 - \frac{5}{12}\, cz_h^2\, \frac{\pi^2 T_C^2}{\pi^2 T^2} \sim 1 - 0.1746\, \frac{T_C^2}{T^2} > 0, \quad \text{if } T > T_C \tag{B.23}$$

where $T_C = 1/\pi^2 z_h^2$ and nothing is problematic.

The vector susceptibility is obatained by eq. (3.9)

$$\chi_V = 2N_f \frac{2\pi^2 T^2}{g_5^2} \tag{B.24}$$

this is same result of hard wall.

C. Other boundary condition

C.1 Hard wall model

We take 5D AdS black hole with the IR cutoff as the dual gravity background to the finite temperature QCD [8, 27, 29]. From the quadratic part of the 5D action eq. (2.1), we obtain the equation of motion for the time component vector meson

$$\left[\partial_z^2 - \frac{1}{z}\partial_z + \frac{\vec{q}^2}{f(z)}\right] V_0(z, \vec{q}) = 0. \tag{C.1}$$

Note that in the above equation, the $q_0 \to 0$ limit is already taken, otherwise the equation will couple to other components. We solve eq. (C.1) in a limit where $\vec{q} \to 0$. The solution is

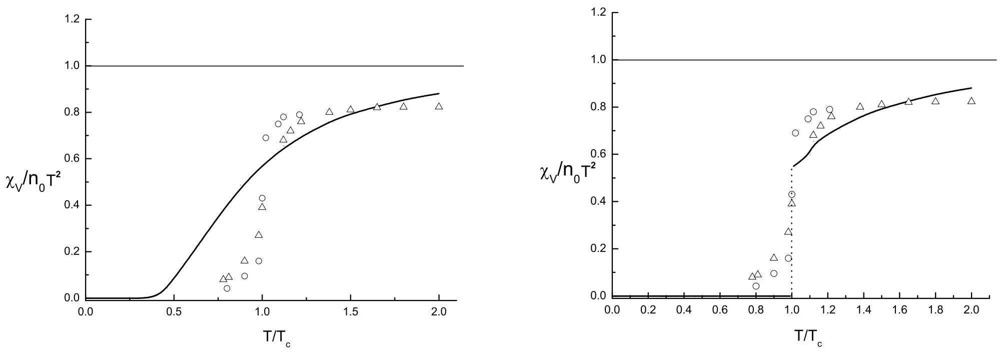
$$V_0 = 1 + a_2 z^2\,. \tag{C.2}$$

Then the vector isospin susceptibility is given by

$$\chi_V(T) = -2N_f \left[\frac{1}{g_5^2}\frac{\partial_z V_0}{z}\right]_{z=0} = -2N_f \frac{2a_2}{g_5^2}. \tag{C.3}$$

The constant dimensionful a_2 is fixed by the boundary conditions (BC). While we have to choose $V_0 = 1$ at UV boundary, we have choices at IR, and the result is sensitive to the IR BC. To see the IR boundary condition dependence, we consider various BCs we can think of: Neumann, Dirichlet, and infalling BCs.

We first consider the Neumann boundary condition at the IR, $\partial_z V_0|_{IR} = 0$. Then $a_2 = 0$ hence $\chi_V = 0$. This is not consistent with the above mentioned results of lattice QCD as well as other calculations.



(a) (b)

Figure 2: χ_V in the soft wall model. (a) without Hawking-Page transition, (b) with Hawking-Page transition; $n_0 \equiv 4N_f\pi^2/g_5^2$. The circles are for the (quenched) lattice QCD results as shown in ref. [14, 18], and the triangles are for full lattice QCD in [19].

For the Dirichlet condition at the horizon, we impose $V_0(z_T) = h$ with constant h and obtain

$$\chi_V(T) = 2N_f\frac{2\pi^2}{g_5^2}(1-h)T^2\,, \tag{C.4}$$

which is proportional to T^2. χ_V/T^2 is a constant which is consistent with high temperature behavior of lattice result [12, 17]. Low temperature behavior can not be extracted from the AdS black hole due to the Hawking-Page transition. That is, thermal AdS replaces the AdS black hole at low temperature. Notice that for the zero frequency and zero momentum, there is no difference between black hole and thermal AdS in the present case.[6] In this case, we need to choose the IR boundary condition such that $h=1$ and $\chi_V=0$ to have a qualitative agreement with the lattice results [12, 17] at low temperature.

In view of the confinement-deconfinement phase transition, these considerations show that $\chi_V(T)/T^2$ jumps from zero, corresponding to the thermal AdS phase, to a constant $(1-h)$, AdS black hole phase, as we increases the temperature.

C.2 Soft wall model

The equation of motion for V_0 in the static-low momentum limit reads

$$\partial_z\left(\frac{1}{z}\mathrm{e}^{-cz^2}\partial_z V_0\right) = 0\,, \tag{C.5}$$

whose solution is given by

$$V_0 = a\mathrm{e}^{cz^2} + b\,. \tag{C.6}$$

[6] This is because the equation of motion given in eq. (C.1) shows that all the f dependent terms goes away in this limit.

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For the Neumann BC, we again get a trivial result. However, for the Dirichlet boundary conditions, $V_0(0) = 1, \;\; V_0(z_T) = h$, we obtain a nontrivial result:

$$\chi_V(T) = 2N_f \frac{2c}{g_5^2} \frac{h-1}{\mathrm{e}^{\tilde{T}_c^2/T^2} - 1} \tag{C.7}$$

where $2N_f$ is from our normalization convention. For the Dirichlet boundary condition, we choose $h = 0$ by hand. For the infalling boundary condition, however, $h = h_0$ must be a very special value. Since this value is independent of temperature and contribute as an overall normalization, we do not proceed to determine it here.

We note here that $\chi_V(\tilde{T}_c) \approx 1.2\tilde{T}_c^2$, where

$$\tilde{T}_c = \sqrt{c}/\pi, \tag{C.8}$$

can be considered as a crossover temperature (assuming that the expression is valid in all temperature regime).

In figure 2, we show the vector isospin susceptibility with and without Hawking-Page transition and compare our results with those from the quenched lattice QCD [18] together with a full one [19]. We refer to [25] for more recent quenched lattice data given above T_c.

In figure 2-(a), we plot the vector isospin susceptibility as a function of $T/\tilde{T}_c$ without Hawking-Page transition. It shows the rapid change of the susceptibility near $\tilde{T}_c$ and at high temperature $\chi_{\mathrm{V}}/(N_f T^2)$ is saturated to the ideal-gas value which may indicate that the transition is rapid but a smooth crossover rather than a phase transition. It is worthwhile to notice that this is rather consistent with a contemporary view of QCD phase diagram based on lattice QCD studies at finite temperature and at zero baryon density. For example, in [28] it is shown that there is a crossover rather than a genuine phase transition from the low-temperature hadron phase to the high-temperature quark-gluon plasma phase. However, we find that our result for the case without HPT, the slope at the transition is rather slow.

Now, we consider the Hawking-Page transition [31], the background metric changes from the AdS black hole to the thermal AdS geometry as we decrease the temperature, inducing a jump in the susceptibility. In fact with thermal AdS, the equation of motion for V_0 is the same with eq. (C.5). All the analysis is the same except that there is no fine cut IR boundary. Fields can penetrate all the way to the deep IR region $z \sim \infty$. For the finiteness $a = 0$ should be imposed. Therefore V_0 is a constant, leading to the zero vector isospin susceptibility at low temperature: $\chi_V = 0$. So in the low temperature regime, χ_V is zero all the way to the phase transition temperature where it jumps to the value given in eq. (C.7), which implies a first order phase transition between low- and high-temperature phases. The sharp transition with HPT might be the large N_c artifact.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200]; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];

E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[2] J. Polchinski and M.J. Strassler, *Hard scattering and gauge/string duality*, *Phys. Rev. Lett.* **88** (2002) 031601 [hep-th/0109174].

[3] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *JHEP* **06** (2002) 043 [hep-th/0205236].

[4] S.J. Brodsky and G.F. de Teramond, *Hadronic spectra and light-front wavefunctions in holographic QCD*, *Phys. Rev. Lett.* **96** (2006) 201601 [hep-ph/0602252].

[5] T. Sakai and S. Sugimoto, *Low energy hadron physics in holographic QCD*, *Prog. Theor. Phys.* **113** (2005) 843 [hep-th/0412141]; *More on a holographic dual of QCD*, *Prog. Theor. Phys.* **114** (2005) 1083 [hep-th/0507073].

[6] J. Erlich, E. Katz, D.T. Son and M.A. Stephanov, *QCD and a holographic model of hadrons*, *Phys. Rev. Lett.* **95** (2005) 261602 [hep-ph/0501128].

[7] L. Da Rold and A. Pomarol, *Chiral symmetry breaking from five dimensional spaces*, *Nucl. Phys.* **B 721** (2005) 79 [hep-ph/0501218].

[8] K. Ghoroku and M. Yahiro, *Holographic model for mesons at finite temperature*, *Phys. Rev.* **D 73** (2006) 125010 [hep-ph/0512289].

[9] O. Aharony, J. Sonnenschein and S. Yankielowicz, *A holographic model of deconfinement and chiral symmetry restoration*, *Ann. Phys. (NY)* **322** (2007) 1420 [hep-th/0604161].

[10] A. Karch, E. Katz, D.T. Son and M.A. Stephanov, *Linear confinement and AdS/QCD*, *Phys. Rev.* **D 74** (2006) 015005 [hep-ph/0602229].

[11] E. Shuryak, *Building a 'holographic dual' to QCD in the* AdS_5*: instantons and confinement*, hep-th/0605219.

[12] S.A. Gottlieb, W. Liu, D. Toussaint, R.L. Renken and R.L. Sugar, *The quark number susceptibility of high temperature QCD*, *Phys. Rev. Lett.* **59** (1987) 2247.

[13] L. McLerran, *A chiral symmetry order parameter, the lattice and nucleosynthesis*, *Phys. Rev.* **D 36** (1987) 3291.

[14] T. Kunihiro, *Quark number susceptibility and fluctuations in the vector channel at high temperatures*, *Phys. Lett.* **B 271** (1991) 395.

[15] P. Chakraborty, M.G. Mustafa and M.H. Thoma, *Quark number susceptibility in hard thermal loop approximation*, *Eur. Phys. J.* **C 23** (2002) 591 [hep-ph/0111022].

[16] M. Harada, Y. Kim, M. Rho and C. Sasaki, *The vector and axial vector susceptibilities and effective degrees of freedom in the vector manifestation*, *Nucl. Phys.* **A 727** (2003) 437 [hep-ph/0207012].

[17] S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken and R.L. Sugar, *Fermion number susceptibility in lattice gauge theory*, *Phys. Rev.* **D 38** (1988) 2888.

[18] R.V. Gavai, J. Potvin and S. Sanielevici, *Quark number susceptibility in quenched quantum chromodynamics*, *Phys. Rev.* **D 40** (1989) 2743.

[19] C.R. Allton et al., *Thermodynamics of two flavor QCD to sixth order in quark chemical potential*, *Phys. Rev.* **D 71** (2005) 054508 [hep-lat/0501030].

[20] R.V. Gavai and S. Gupta, *Quark number susceptibilities and the Wroblewski parameter*, *Eur. Phys. J.* **C 43** (2005) 31 [hep-ph/0502198].

[21] S. Ejiri, F. Karsch and K. Redlich, *Hadronic fluctuations at the QCD phase transition*, *Phys. Lett.* **B 633** (2006) 275 [hep-ph/0509051].

[22] A. Hietanen and K. Rummukainen, *Quark number susceptibility of high temperature and finite density QCD*, PoS(LATTICE 2007)192 [arXiv:0710.5058].

[23] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[24] C.P. Herzog, *A holographic prediction of the deconfinement temperature*, *Phys. Rev. Lett.* **98** (2007) 091601 [hep-th/0608151].

[25] R.V. Gavai and S. Gupta, *The continuum limit of quark number susceptibilities*, *Phys. Rev.* **D 65** (2002) 094515 [hep-lat/0202006].

[26] G. Policastro, D.T. Son and A.O. Starinets, *From AdS/CFT correspondence to hydrodynamics*, *JHEP* **09** (2002) 043 [hep-th/0205052].

[27] H. Boschi-Filho, N.R.F. Braga and C.N. Ferreira, *Heavy quark potential at finite temperature from gauge/string duality*, *Phys. Rev.* **D 74** (2006) 086001 [hep-th/0607038].

[28] Y. Aoki et al, *The order of the quantum chromodynamics transition predicted by the standard model of particle physics*, *Nature* **443** (2006) 675 [hep-lat/0611014];
MILC collaboration, C. Bernard et al., *QCD thermodynamics with three flavors of improved staggered quarks*, *Phys. Rev.* **D 71** (2005) 034504 [hep-lat/0405029];
U.M. Heller, *Recent progress in finite temperature lattice QCD*, plenary talk at *XXIV*$^{\text{th}}$ *International Symposium on Lattice Field Theory (Lattice* 2006*)*, Tucson Arizona U.S.A. July 23–28 2006 [PoS(LAT2006)011].

[29] O. Andreev and V.I.¿Zakharov, *The spatial string tension, thermal phase transition and AdS/QCD*, *Phys. Lett.* **B 645** (2007) 437 [hep-ph/0607026].

[30] O. Andreev, *1/q^2 corrections and gauge/string duality*, *Phys. Rev.* **D 73** (2006) 107901 [hep-th/0603170].

[31] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, *Commun. Math. Phys.* **87** (1983) 577.

[32] O. Andreev and V.I. Zakharov, *Heavy-quark potentials and AdS/QCD*, *Phys. Rev.* **D 74** (2006) 025023 [hep-ph/0604204].

[33] H.A. Weldon, *Particles and holes*, *Physica* **A 158** (1989) 169;
S.A. Gottlieb et al., *Thermodynamics of lattice QCD with two light quark flavors on a* $16^3 \times 8$ *lattice.* 2, *Phys. Rev.* **D 55** (1997) 6852 [hep-lat/9612020];
H.A. Weldon, *New mesons in the chirally symmetric plasma*, hep-ph/9810238.